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LOCAL SCALING OF TIME IN HAMILTONIAN PATH INTEGRATION

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ABSTRACT

Inspired by the usefulness of local scaling of time in the path integral formalism, we introduce a new kind of hamiltonian path integral in this paper. A special case of this new type of path integral has been earlier found useful in formulating a scheme of hamiltonian path integral quantization in arbitrary coordinates. This scheme has the unique feature that quantization in arbitrary co-ordinates requires hamiltonian path integral to be set up in terms of the classical hamiltonian only, without addition of any adhoc $O(\hbar^2)$terms. In this paper we further study the properties of hamiltonian path integrals in arbitrary co-ordinates with and without local scaling of time and obtain the Schrodinger equation implied by the hamiltonian path integrals. As a simple illustrative example of quantization in arbitrary coordinates and of exact path integration we apply the results obtained to the case of Coulomb problem in two dimensions.

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1. Introduction

The method of path integration [1] is an attractive way of obtaining the quantum mechanical amplitudes. There are two main approaches to path integrals. One of them is to define them rigorously in terms of a class of paths and measures defined on the class of paths [2]. The other approach is more widely practiced. For reviews of the path integral method and its applications we refer the reader to references 3-9. The so called 'time slicing' method contains, essentially, nothing more than the observation that the propagator, being the matrix element of $\exp[-itH/\hbar]$,can be written as an $N \to \infty$ limit of an $N-$ fold interal of the matrix elements of $\exp[-itH/\hbar]$. The latter approach, in spite of its shortcomings in defining the paths and measure rigorously, continues to be very useful. If the path integral is to be treated as an alternative procedure for quantization, one must try to recover at least all the exactly solvable problems of non-relativistic quantum mechanics. For a long time, apart from the quadratic action problems the only non-gaussian potential problem which could be solved exactly by the the path integral method was the $1/r^2$ potential problem[11]. While Duru and Kleinert[12] succeeded in doing the exact path integration for the Green function of the hydrogen atom problem using the Kustanheimo-Stiefel transformation[13], the manipulations used in the original paper were formal and lacked mathematical justification. For the H- atom problem a complete treatment was given by Ho and Inomata[14]. For many other potential problems exact path integartion could be completed in the next few years. This has been possible thanks to the idea of using local scaling of time in the path integral approach. Change of variable followed by local scaling of time or addition of new degrees of freedom have since been used by many authors to give path integral solutions for many potential problems of quantum mechanics. The Coulomb problem in two dimensions[15] and the radial Coulomb problem[16], the Coulomb field with magnetic charges[17] and the Coulomb potential with the Bohm Aharonov potential[18] and the non-relativistic dyonium problem [19] are some of the Coloumb field related problems which have been solved exactly in the path integral formalism. Important potential problems such as the Morse oscillator [20], the Rosen-Morse potential [21], the Poschl-Teller oscillator [21], the Hartmann potential [22], the Hulthen potential[24], the $\delta-$ potential[25], the square well [26] and many other potential problems [27] have been solved exactly by the path integral method.

Motivated by the importance of local scaling of time in the path integral formalism, in this paper we introduce a new kind of hamiltonian path intergral which is termed as hamiltonian path integral with local scaling of time. This new path integral is defined in terms of canonical form of the path integral introduced by one of us earlier [28]. A special case of the new path integral to be defined here has been found suitable for hamiltonian path integral quantization scheme in arbitrary co- ordinates [29]. These papers will, hereafter, be referred to as papers I and II respectively. *The quantization scheme of paper II has the merit that, irrespective of the co-ordinates chosen, it is formulated in terms of classical Hamiltonian only.* This may be contrasted with existing Hamiltonian path integral quantization methods in which it becomes necessary to add the $O(\hbar^2)$ effective potential to the classical Hamiltonian in non-cartesian co-ordinates[30].

The new path integral intorduvced in this paper can be considered to be a generalization of the path integral defined in the paper I. In this paper we study in detail the two hamiltonian path integrals of paper I and its generalization defined here and obtain several important results about them. Our discussion of local scaling of time differs from other treatments existing in the literature [31]. In the existing treatments, quantum mechanics is assumed and the path integral with local scaling of time is derived. However, we directly define the canonical path integral with local scaling of time. We then obtain a connection with quantum mechanics by deriving the time development equation implied by the assumed skeletonised form of the path integral with scaling. For path integrals, without local scaling of time, the method of obtaining the Schrodinger equation is well-known. However, for the path integrals with local rescaling of time our approach is the first of its kind for deriving the time development equation.

The plan of this paper is as follows. As a preparation to study of hamiltonian path integration with non-trivial scaling, in Sec. 2 we first recall the definition of hamiltonian path integral of paper I and prove some of its properties needed later. The Schrodinger equation and the ordering implied by the hamiltonian path integral are derived. The central results of this paper on properties of the hamiltonian path integral with nontrivial scaling are derived in Sec.3. As applications of the results in this section, we recover the quantization scheme of paper II. The two dimensional Coulomb problem in parabolic coordinates is discussed as an example of quantization in arbitrary coordinates. This example is relatively simple, setting up path integral quan-

tization in arbitrary co-ordinates immediately leads to exact answer for the propagator. A detailed solution for many other potential problems can be given using results derived in this paper and will be taken up in a separate publication.

2. The hamiltonian path integral without scaling

2.1 Introduction : In this section shall define hamiltonian path integral without scaling and discuss its properties as a preparation to study of hamiltonian path integartion with local scaling of time in Sec. 3.*We shall assume throughout this paper that the Hamiltonian is independent of time and that it is quadratic in momenta.* To simplify our discussion we shall not keep terms linear in momenta, for these do not require any technique other than that developed in our paper.

In the time slicing approach the basic building block of the path integral is the approximate expression of the propagator for short times, which will be called the short time propagator (STP). *If the path integral is to be used as an independent scheme of quantization, the act of quantization consists precisely in choosing the STP and using it to do summation over histories.* Let us generally denote the STP as $(qt|q_0t_0)$. As it stands it is merely a generalized function of the indicated variables and, for short times, approximates the matrix element of a unitary transformation. For the Lagrangians of type

$$L = \frac{1}{2}g_{ij}(q)\frac{dq^i}{dt}\frac{dq^j}{dt} - V(q) \tag{1}$$

an important form of STP is the Van Vleck- Pauli- Morette form of short time approximation [32,33,34] given by

$$(qt|q_0t_0) = (2\pi i\hbar)^{-n/2}(g(q)g(q_0))^{1/4}\sqrt{D}\exp[iS(qt,q_0t_0)/\hbar] \tag{2}$$

Here n is the number of degrees of freedom, S is the classical action for the trajectory passing through the points (q,t) and (q_0,t_0), $g = \det(g_{ij})$ and D is the Van Vleck determinant given by

$$D = \det\left(-\frac{\partial^2 S}{\partial q^i \partial q_0^j}\right) \tag{3}$$

This Van Vleck- Pauli- Morette (VPM) form has the following remarkable properties [3,35] which will be generalized when setting up the hamiltonian path integration.

1. In those cases where the quantization can be done using canonical methods, the semi-classical limit of the exact propagatoris seen to be just the VPM formula, even for finite $t - t_0$. In some special cases the semi-classical limit is already the exact propagator [35].

2. If q_2 integral is computed in the stationary phase approximation the VPM form of STP obeys the semi-group property

$$\int (q_3 t_3 | q_2 t_2) \rho(q_2) d^n q_2 (q_2 t_2 | q_1 t_1) = (q_3 t_3 | q_1 t_1) \qquad (4)$$

3. For $t - t_0 \rightarrow 0$ it obeys the boundary condition

$$\lim_{t \to t_0} (qt | q_0 t_0) = g(q)^{-1/2} \delta(q - q_0) \qquad (5)$$

4. Under arbitrary co-ordinate transformations the expression for STP remains unchanged because both the classical trajectory and the action along it are invariant concepts. Moreover the normalization (5) is invariant under arbitrary co-ordinate transformations.

5. The action S is the generating function of canonical transformation connecting the co-ordinates and momenta at times t to those at times t_0 . It satisfies the Hamilton Jacobi equation

$$\frac{\partial S}{\partial t} + H(q, \frac{\partial S}{\partial q}) = 0 \qquad (6)$$

The Lagrangian form of path integration in arbitrary co-ordinates can be obtained by using the VPM form of STP to do summation over histories. This gives the correct quantization rules in arbitrary co-ordinates when the classical Lagrangian is used [36].

 2.2 The Hamiltonian Path Integral : The VPM form of the short time approximation to the propagator shows the crucial significance of the classical trajectory and the classical action. Using this anology we shall now define the STP for the hamiltonian path integral as a two step process.

 Let γ_1 and γ_2 be two classical trajectories with boundary conditions as indicated below.

$$\gamma_1 : \tau \rightarrow (\tilde{q}(\tau), \tilde{p}(\tau)), \quad \text{for} \quad t_0 \leq \tau \leq t_1 \tag{7}$$

$$\tilde{q}(t_0) = q_0, \quad \tilde{p}(t_1) = p_1 \tag{8}$$

$$\gamma_2 : \tau \rightarrow (\tilde{\tilde{q}}(\tau), \tilde{\tilde{p}}(\tau)), \quad \text{for} \quad t_1 \leq \tau \leq t \tag{9}$$

$$\tilde{\tilde{p}}(t_1) = p_1, \quad \tilde{\tilde{q}}(t) = q \tag{10}$$

$$(qt|p_1t_1) = (2\pi\hbar)^{-n/2}\sqrt{D_{++}} \exp[iS_{++}(qt, p_1t_1)/\hbar] \tag{11}$$

$$(p_1t_1|q_0t_0) = (2\pi\hbar)^{-n/2}\sqrt{D_{--}} \exp[iS_{--}(p_1t_1, q_0t_0)/\hbar] \tag{12}$$

where

$$D_{++} = \det\left(\frac{\partial^2 S_{++}}{\partial q^i \partial p_{1j}}\right) \tag{13}$$

$$D_{--} = \det\left(\frac{\partial^2 S_{--}}{\partial q^i_j \partial p_{1j}}\right) \tag{14}$$

and S_{++}, S_{--} are Legendre transforms of the classical action along the two trajectories,

$$S_{++}(qt, p_1t_1) = p_{1i}\tilde{\tilde{q}}_1^{\ i} + \int_{\gamma_2} (\tilde{\tilde{p}}_i d\tilde{\tilde{q}}^i(\tau) - H(\tilde{\tilde{q}}(\tau), \tilde{\tilde{p}}(\tau)))d\tau \tag{15}$$

$$S_{--}(q_0t_0, p_1t_1) = -p_{1i}\tilde{q}_1^{\ i} + \int_{\gamma_1} (\tilde{p}_i d\tilde{q}^i - H(\tilde{q}(\tau), \tilde{p}(\tau))) \, d\tau \tag{16}$$

where

$$\tilde{q}_1 \equiv \tilde{q}(t_1), \qquad \tilde{\tilde{q}}_1 \equiv \tilde{\tilde{q}}(t_1) \tag{17}$$

In (11) to (16) it is understood that the independent variables are those explicitly shown on the left hand side. Also the right hand sides are supposed to have been expressed in terms of these variables using canonical transformations, or equivalently, using the equations for the classical trajectories (7) and (9) as functions of boundary conditions.

The canonical STP is now defined as

$$(qt\|q_0t_0) = \frac{1}{\sqrt{\rho(q)\rho(q_0)}} \int d^n p_1 \, (qt|p_1t_1) \, (p_1t_1|q_0t_0) \tag{18}$$

where we have introduced the factor $\sqrt{\rho(q)\rho(q_0)}$ in the definition of STP so that it satisfies the semi-group property in the stationary phase approximation with respect to the measure $\rho(q)d^n q$.

$$\int (q_3 t_3 \| q_2 t_2)\rho(q_2)d^n q_2 (q_2 t_2 \| q_1 t_1) \approx (q_3 t_3 \| q_1 t_1) \tag{19}$$

Several remarks are now in order concerning the above definition.

1. The functions S_{++} and S_{--} obey the corresponding Hamilton Jacobi equations.

$$\frac{\partial S_{++}}{\partial t} + H\left(q, \frac{\partial S_{++}}{\partial q}\right) = 0 \tag{20}$$

$$\frac{\partial S_{--}}{\partial t_0} + H\left(\frac{\partial S_{--}}{\partial p}, p\right) = 0 \tag{21}$$

2. The functions S_{++} and S_{--} always exist for any given $H(q,p)$ because the variables (q, p_1), and also (q_0, p_1), always form independent sets of variables, at least for small enough intervals $t - t_1$ and $t_1 - t_0$. The same cannot be said for the pairs $(p, p_1), (q_2, q), (q, q_1)$ or (p_1, p_n). If, for example, $H(q, p)$ is a function of $p's$ alone, the momenta are constants of motion and, therefore, the values of momenta at different times cannot be independent.

3. The canonical STP satisfies the boundary condition

$$\lim_{t \to t_0} (qt \| q_0 t_0) = (\rho(q))^{-1/2} \delta^{(n)}(q - q_0) \tag{22}$$

Such a STP, after doing a summation over histories, will give rise to a unitary transformation on a Hilbert space with measure $\rho(q)$.

4. The expression (18) becomes identical with the VPM form if the p_1- integrals are evaluated in the stationary phase approximation.

5. The expressions $(qt \| p_1 t_1)$ and $(p_1 t_1 \| q_0 t_0)$ correspond to short time approximations, respectively, of the quantum mechanical amplitudes.

$$< q | \exp[-i H_{op}(t - t_1)/\hbar] | p_1 > \tag{23}$$

and

$$< p_1 | \exp[-i H_{op}(t_1 - t_0)/\hbar] | q_0 > \tag{24}$$

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We shall now discuss the properties of the hamiltonian path integral of paper I obtained by summing over histories with (18) inserted for STP. We shall derive the Schrodinger equation satisfied by the hamiltonian path integral and the operator ordering rule which relates the operator H_{op} appearing in the above expressions to the function $H(q, p)$ used to set up the STP.

The STP, as defined by (7)-(18) above, will hereafter be called the *canonical STP* for the Hamiltonian function $H(q, p)$. It gives rise to a path integral in the limit $N \to \infty$ of

$$K^{(N)}[H, \rho](qt; q_0 t_0) = \int \prod_{k=1}^{N-1} \rho(q_k) dq_k \prod_{j=0}^{N-1} (q_{J+1}\epsilon \| q_j 0) \qquad (25)$$

where $\epsilon = (t - t_0)/N$ and $q_N \equiv q$. The path integral so obtained will be referred to as the *first hamiltonianl path integral, or HPI1,* for the Hamiltonian function $H(q, p)$ and measure $\rho(q)$. It will be denoted by $K[H, \rho](qt; q_0 t_0)$. Thus

$$K[H, \rho](qt; q_0 t_0) = \lim_{N \to \infty} K^{(N)}[H, \rho](qt; q_0 t_0) \qquad (26)$$

2.3 Elementary properties of the hamiltonian path integral :

(a) It is useful to note here that, if a constant E is added to the hamiltonian, the HPI1 changes by a phase factor

$$K[H - E, \rho](qt; q_0 t_0) = \exp[iE(t - t_0)/\hbar] K[H, \rho](qt; q_0 t_0) \qquad (27)$$

(b) The dependence of the HPI1 $K[H, \rho](qt; q_0 t_0)$ on the measure ρ is very simple. In fact we have the following relations

$$
\begin{aligned}
\sqrt{\tilde{\rho}(q)\tilde{\rho}(q_0)} K[H, \tilde{\rho}](qt; q_0 t_0) &= \sqrt{\rho(q)\rho(q_0)} K[H, \rho](qt; q_0 t_0) \\
&= \text{independent of } \rho \qquad (28)
\end{aligned}
$$

To prove the above statement, observe that in the expression for $K^{(N)}[H, \rho]$ the factors $\rho(q_1)...\rho(q_{N-1})$ coming fromthe STP cancel with those from the integration measure leaving the factor $\sqrt{\rho(q)\rho(q_0)}$ coming from ρ on the two end points only. Thus $K^{(N)}[H, \rho]$ and $K^{(N)}[H, \tilde{\rho}]$ constructed for different measures satisfy a relation similar to (28) and the proposition follows by taking the lim $N \to \infty$.

(c) In the cartesian coordinates $K[H, \rho]$ gives the correct quantum mechanical propagator.

2.4 Discretized form for the hamiltonian path integral : The hamiltonian path integral $K[H, \rho]$ defined by (25) and (26), represents the matrix element of a unitary transformation. Therefore the wave function propagated by it satisfy a certain Schrodinger equation. In order to derive the Schrodinger equation we need an explicit expression for the STP $(qt\|q_0t_0)$.

Proposition 2.1 : Let $\epsilon_1 = t - t_1$ and $\epsilon_2 = t - t_0$ be small. Then

$$S_{--}(p_1t_1, q_0t_0) = -p_{1i}q_0^i - H(q_0, p_1)\epsilon_1 + O(\epsilon_1^2) \tag{29}$$

$$S_{++}(qt, p_1t_1) = p_{1i}q^i - H(q, p_1)\epsilon_2 + O(\epsilon_2^2) \tag{30}$$

The proof is fairly straightforward computation using the regular behavior of the classical trajectories γ_1 and γ_2 and formulas such as

$$\int_{x_1}^{x_2} f(x)dx = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} f^{(n)}(x_1)(x_2 - x_1)^{n+1} \tag{31}$$

$$\int_{x_1}^{x_2} f(x)dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!} f^{(n)}(x)(x_2 - x_1)^{n+1} \tag{32}$$

Applying this result to (13)-(16) we immediately get

Proposition 2.2: For small ϵ_1 and ϵ_2 we have

$$(q\epsilon\|q_00) = (\rho(q)\rho(q_0))^{1/2} \int \frac{d^np}{(2\pi\hbar)^n} \exp[ip_i(q^i - q_0^i)/\hbar] \times$$

$$\exp[-iH(q, p)\epsilon/\hbar - iH(q_0, p)\epsilon/\hbar] \left[1 - \frac{\epsilon_1}{2}\frac{\partial^2 H(q, p)}{\partial q^i \partial p_i} + \frac{\epsilon_2}{2}\frac{\partial^2 H(q_0, p)}{\partial q_0^i \partial p_i} + \cdots\right] \tag{33}$$

This is the form which will be useful for later aplications. For hamiltonian quadratic in momenta, the hamiltonian STP can be reduced to a standard Lagrangian form of STP, such as the mid point form, by carrying out the p- integrations and using the McLaughlin Schulman trick [3]6. It is fairly easy and straight forward to work out and the details can be found in ref 37. For the present purpose we write it in a slightly different form valid for short times From now onwards we shall choose t_1 to be midway between t_0 and t. Thus we shall take

$$t - t_1 = \epsilon_2 = \epsilon_1 = t - t_0 = \epsilon \tag{34}$$

The reason for this choice is that it gives a self- adjoint (symmetric) Hamiltonian operator, though for H quadratic in momenta it is not essential to

make this choice. Neglecting $O(\epsilon^2)$ terms we get

$$(q\epsilon\|q_0 0)$$

$$= (\rho(q)\rho(q_0))^{1/2} \int \frac{d^n p}{(2\pi\hbar)^n} \exp[ip_i(q^i - q_0^i)/\hbar] \times$$

$$\left[1 - \frac{i\epsilon}{\hbar}H(q,p) - \frac{i\epsilon}{\hbar}H(q_0,p) - \frac{\epsilon}{2}\frac{\partial^2 H(q,p)}{\partial q^i \partial p_i} + \frac{\epsilon}{2}\frac{\partial^2 H(q_0,p)}{\partial q_0^i \partial p_i} + ...\right] \quad (35)$$

2.5 Schrodinger equation for the hamiltonian path integral. Let us consider the Hilbert space of wave functions $\psi(q)$ defined on the configuration space with measure $\rho(q)d^n q$. In other words, $\psi \in L^2(R^n, \rho(q)d^n q)$ if $\int |\psi(q)|^2 \rho(q)d^n q \leq \infty$. Let ψ be propagated by the canonical STP :

$$\psi(q,t) = \int K[H,\rho](qt;q_0 t_0)\psi(q_0,t_0)\rho(q_0)d^n q_0 \quad (36)$$

The equation satisfied by $\psi(q,t)$ is given by the following proposition.

Proposition 2.3 : $\psi(q,t)$ defined by (36) satisfies

$$i\hbar\frac{\partial\psi}{\partial t}(q,t) = H_{op}\psi(q,t) \quad (37)$$

where the operator H_{op}, defined on $L^2(R^n, \rho(q)d^n q)$, is obtained by the following prescription:

1. Define

$$H_1(q,p) = \left[H(q,p) - \frac{i\hbar}{2}\frac{\partial^2 H}{\partial q^i \partial p_i}\right] \quad (38)$$

2. Bring all q' s in H_1 to the left and all p' s to the right. Replace each p_i in H_1, so ordered, by $-i\hbar\partial/\partial q^i$. This makes H_1 a differential operator, say, $X(q,\partial/\partial q)$, acting on functions of q.

3. Next bring all q' s in H_1 to the right and all p' s to the left. Replace each p_i in H_1, so ordered, by $-i\hbar\partial/\partial q^i$. This makes H_1 a differential operator, say, $Y(q,\partial/\partial q)$, acting on functions of q.

4. Define $\widehat{H}$ by symmetrization,

$$\widehat{H} = X\left(q,\frac{\partial}{\partial q}\right) + Y\left(q,\frac{\partial}{\partial q}\right) \quad (39)$$

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5. Finally, define H_{op} by

$$H_{op} = \rho^{-1/2}\widehat{H}\rho^{1/2} \tag{40}$$

The proof of this prescription is based on the formula (35) and the fact that, with $x^i = (q - q_0)^i$,

$$\int \rho(q_0)d^n q_0 \int \frac{d^n p}{(2\pi\hbar)^n} \exp(ip_i x^i/\hbar) \frac{f(q)p_j g(q_0)}{\sqrt{\rho(q_0)\rho(q)}} \psi(q_0)$$

$$= f(q)(\rho(q))^{-1/2} \int d^n q_0 (\rho(q_0))^{1/2}\psi(q_0)[-i\hbar\frac{\partial}{\partial x^j}]\delta^n(x)$$

$$= f(q)(\rho(q))^{-1/2}(-i\hbar\frac{\partial}{\partial q_0^j})((\rho(q_0))^{1/2}g(q_0)\psi(q_0))\Big|_{q_0=q}$$

$$= (f(q)\hat{p}_j g(q))\psi(q) \tag{41}$$

The expression $f(q)p_j g(q_0)$ inside the above integral becomes equal to the ordered operator $f(q)\hat{p}_j g(q)$. The above rules are now a direct generalizations of these formulas to arbitrary functions $H(q,p)$, at least, to the functions which are polynomials in p. For real $H(q,p)$, the operator H_{op} defined by the proposition 2.3 is also hermitian on $L(R^n, \rho(q)d^n q)$

An important case of interest is the Schrodinger equation when the hamiltonian function is given by

$$H_0 = \frac{1}{2m}g^{ij}p_i p_j \tag{42}$$

where g^{ij} is the contravariant metric tensor (matrix inverse of g_{ij}). In this case we have

$$\widehat{H_0} = \frac{1}{2m}\frac{\partial}{\partial q^i}g^{ij}\frac{\partial}{\partial q^j} \tag{43}$$

$$(H_0)_{op} = \frac{1}{\sqrt{\rho}}\widehat{H_0}\sqrt{\rho} \tag{44}$$

$$= -\frac{\hbar^2}{2m}\frac{1}{\sqrt{\rho}}\frac{\partial}{\partial q^i}g^{ij}\left(\frac{\partial}{\partial q^j}\sqrt{\rho}\right) \tag{45}$$

$$= -\frac{\hbar^2}{2m}\Delta_\rho + U(\sqrt{g}) \tag{46}$$

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Here Δ_ρ is self-adjoint Laplace-Beltrami operator on $L^2(R^n \rho(q) d^n q)$ related to the normal Laplace-Beltrami operator on $L^2(R^n \sqrt{g} d^n q)$ as

$$\Delta_\rho = \rho^{-1/2} g^{1/4} \Delta g^{-1/4} \rho^{1/2} \qquad (47)$$

$$\Delta = g^{-1/2} \frac{\partial}{\partial q^i} (g^{1/2} g^{ij} \frac{\partial}{\partial q^j}) \qquad (48)$$

with

$$U(h) = -\frac{\hbar^2}{8m} [g^{ij} (\ln h)_{,i} (\ln h)_{,j} + 2(g^{ij} (\ln h)_{,i})_{,j}] \qquad (49)$$

$$= -\frac{\hbar^2}{2m} [g^{ij} (\sqrt{h}_{,ij} / \sqrt{h} + g_i^{ij} (\sqrt{h})_{,j} / \sqrt{h}] \qquad (50)$$

The result (46) shows that HPI1 $K[H, \sqrt{g}]$ for the classical hamiltonian H_0, set up as in paper I, gives correct Schrodinger equation only in the cartesian co-ordinates and in general $U(\sqrt{g})$, which is of the order $\hbar^2$, must be subtracted from the classical hamiltonian to get correct quantization schme.In this respect HPI1 is no better than any other existing hamiltonian path integral scheme. However, as we shall see in the next section, HPI1 can be used to define a second kind of path integral which involves suitable scaling of time and which with the classical hamiltonian as input gives rise to the correct Schrodinger equation in every co-ordinate system.

3. Hamiltonian path integral with scaling

Local scaling of time has proved to be a useful technique in completing exact path integration for several potential problems. Just as in mathemaics good results are frequently turned into definitions, we use an expression appearing in the scaling relation to define a hamiltonian path integral of second kind, one with local scaling of time. A special case of this path integral already used in paper II which, with the classical hamiltonian and a suitable scaling function, was found to give rise to the correct Schrodinger equaion in every set of co-ordinate system.

We introduce a hamiltonian path integral $\mathcal{K}[H, \rho, \alpha]$ with scaling by means of the equation

$$\mathcal{K}[H, \rho, \alpha](qt, q_0 0)$$
$$\equiv \sqrt{\alpha(q)\alpha(q_0)} \int \frac{dE}{2\pi\hbar} \exp(-iEt/\hbar) \int d\sigma K[\alpha(H - E), \rho](q\sigma, q_0 0) \quad (51)$$

This path integral will be called the second hamiltonian path integral, or HPI2. It depends on the hamiltonian $H(q, p)$, integration measure ρ and scaling function $\alpha(q)$. In the right hand side of the above equation $K[\alpha(H-E), \rho]$ is the HPI1 obtained, after summation over histories, from the canonical STP corresponding to the function $\alpha(H(q, p) - E)$. The above definition is such that for $\alpha(q) =$ a constant, the HPI2 $\mathcal{K}[H, \rho, \alpha]$ coincides with HPI1 $K[H, \rho]$. We have found it useful to not to fix form of $\alpha(q)$ and $\rho(q)$ at this stage just as the form of hamiltonian function $H(q, p)$ is not fixed while defining the path integral. As we shall see below, the path integral HPI2 generalizes HPI1 of paper I and HPI2 reduces to HPI1 when $\alpha(q)$ is indepndent of q.

The central results of this paper on local scaling of time in canonical path integral are given in the propositions 3.1 to 3.3 given below.

Proposition 3.1: The Schrodinger equation satisfied by the wave function propagated by the HPI2, $\mathcal{K}[H, \rho, \alpha]$ for $H_0 = (1/2m)g^{ij}p_i p_j$ is

$$i\hbar \frac{\partial \psi}{\partial t} = \rho^{-1/2}(\alpha^{-1/2}\widehat{\alpha H_0}\alpha^{-1/2})\rho^{1/2} \ \psi \qquad (52)$$

and $\mathcal{K}[H, \rho, \alpha]$ has the normalization $\rho^{-1}\delta(q - q_0)$. In the above equation $\widehat{\alpha H_0}$ is the operator obtained from the function $\alpha(q)H_0(q, p)$ by applying rules 1 to 4 given in proposition 2.3.

We shall prove the proposition for $\rho = 1$; there will be no $\sqrt{\rho(q_k)\rho(q_{k+1})}$ factors which occur for normalization corresponding to $\int \rho d^n q$ in (33). The proof for an arbitrary ρ does not involve any additional difficulty.

We now construct the canonical STP from (33) and set up its $N-$ fold integration; there are $2N - 1$ intermediate points in the interval $[0, \sigma]$ and and corresponding variables are $p_0, q_1, p_1, ..., q_{N-1}, p_{N-1}$. We call q_N. the end point q.

$$K^{(N)}[\alpha(H - E), 1](q\sigma, q_0 0)$$
$$= \int dp_0 \prod_{r=1}^{N-1} \left(\int \frac{dq_r dp_r}{(2\pi\hbar)^n} \right) \exp\left[\frac{i}{\hbar} \sum_{j=0}^{N-1} S_E(q_{j+1}, p_j, q_k; \sigma) \right] \times$$
$$\prod_{k=0}^{N-1} \left[1 - \frac{\sigma}{4N}p_{ki}\frac{\partial(\alpha_{k+1}g_{k+1}^{ij})}{\partial q_k^j} + \frac{\sigma}{4N}p_{ki}\frac{\partial(\alpha_k g_k^{ij})}{\partial q_k^j} \right] \qquad (53)$$

where

$$S_E(q_{k+1}, p_k, q_k; \sigma)$$
$$= p_{ki}(q_{k+1}^i - q_k^i) - \frac{\sigma}{2N}(\alpha_{k+1} H_0(q_{k+1}, p_k) + \alpha_k H_0(q_k, p_k) - (\alpha_{k+1} + \alpha_k)E) \quad (54)$$

and

$$\alpha_k = \alpha(q_k), \quad g_k^{ij} = g^{ij}(q_k) \quad q_N = q \qquad \text{etc.} \qquad (55)$$

and we hope that the pseudo-time lattice index k will not be confused with the indices i and j referring to the different co-ordinates. In any case all summations and products are displayed explicitly.

Substituting the expression (53) for $K^{(N)} in$,

$$\mathcal{K}[H, 1, \alpha](qt, q_0 0)$$

$$\equiv \lim_{N \to \infty} \sqrt{\alpha(q)\alpha(q_0)} \int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar} \exp(-iEt/\hbar) \int_0^{\infty} d\sigma K^{(N)}[\alpha(H - E), 1](q\sigma, q_0 0)$$
$$(56)$$

we first evaluate the E integral which gives us the delta function

$$\delta\left(\frac{\sigma}{2N}(\alpha_0 + 2\sum_{k=1}^{N-1} \alpha_k + \alpha) - t\right) \qquad (57)$$

where $\alpha_N = \alpha(q) = \alpha, \alpha_0 = \alpha(q_0)$. This enables us to do the σ integral. Since all α' s are positive, the $\delta-$ function definitely contributes for some σ in the range $(0, \infty)$. In fact it contributes a factor

$$2N\left(\alpha_0 + \alpha_N + 2\sum_{k=1}^{N-1} \alpha_k \alpha_k\right)^{-1} \equiv 2N/F(q_0, q_1, ..., q_N) \qquad (58)$$

at

$$\sigma = 2Nt/F(q_0, q_1, ..., q_N) \qquad (59)$$

For later use we note that when all the arguments $q_0, ..., q_N$ are equal $F/2N$ becomes equal to $\alpha(q)$, i.e.,

$$F(q, q, ..., q)/2N = \alpha(q) \qquad (60)$$

Performing the σ integral with the help of the $\delta-$ function in (57) as explained above gives

$$\mathcal{K}[H,1,\alpha](qt,q_00)$$

$$= \lim_{N\to\infty} \sqrt{\alpha(q)\alpha(q_0)} \int dp_0 \prod_{r=1}^{N-1} \left(\int \frac{dq_r dp_r}{(2\pi\hbar)^n} \right) (\frac{2N}{F}) \times$$

$$\exp\Big[\frac{2i}{\hbar} \sum_{j=0}^{N-1} S(q_{j+1},p_j,q_j; \frac{2Nt}{F}\Big] \prod_{k=0}^{N-1} \Big[1 - \frac{t}{2F}p_{ki}\frac{\partial(\alpha_{k+1}g_{k+1}^{ij})}{\partial q_{k+1}^j} + \frac{t}{2F}p_{ki}\frac{\partial(\alpha_k g_k^{ij})}{\partial q_k^j}\Big]$$

$$(61)$$

Our aim is to find the Schrodinger equation satisfied by the wave function propagated by $\mathcal{K}[H,1,\alpha]$. For this purpose it is sufficient to retain the terms of order t and neglect all other higher order terms. We must emphasize here that we cannot take σ small and replace HPI1 in (51) by a *single* STP. For small t We write the factors under the integral sign in (61 as

$$\frac{2N}{F} \exp\Big[\frac{i}{\hbar} \sum_{j=1}^{N-1} p_j(q_{j+1} - q_j)\Big] \times$$

$$\Big\{1 - \frac{it}{\hbar F} \sum_{m=0}^{N-1} (\alpha_{m+1}H_0(q_{m+1},p_m) + \alpha_m H_0(q_m,p_m))\Big\} \times$$

$$\Big\{1 - \frac{t}{2F} \sum_{m=0}^{N-1} \Big[\frac{\partial}{\partial q_{m+1}^i}\Big(\alpha_{m+1}g_{m+1}^{ij}\Big) p_{mj} - \frac{\partial}{\partial q_m^i}\Big(\alpha_m g_m^{ij}\Big) p_{mj}\Big]\Big\}$$

$$(62)$$

For the terms independent of t, i.e., in the leading term, 1, coming from the product of the expressions inside the curly brackets, there is no $p-$ dependence at all and the p integrations give rise to a product of $\delta-$ functions in differences $q_{k+1} - q_k$ which can be used to perform all $q_k, k = 1, 2, ..., q_{N-1}$, integrals. Noting (60) we get

$$\sqrt{\alpha(q)\alpha(q_0)}\Big[\frac{2N}{F}\Big]\delta^n(q - q_0) = \delta^n(q - q_0) \qquad (63)$$

Next we consider the terms of order t in (62). These terms are typically of the form

$$\sqrt{\alpha(q)\alpha(q_0)} \int dp_0 \left(\prod_{r=1}^{N-1} \int \frac{dq_r dp_r}{(2\pi\hbar)^n}\right) \left(-\frac{2iN}{\hbar F^2}\right) \times$$

$$\exp\Big[\frac{i}{\hbar} \sum_{j=1}^{N-1} p_k(q_{j+1} - q_j)\Big] J(q_{m+1},p_m,q_m) \qquad (64)$$

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with J as an abbreviation for the mth term obtained after multiplying out the curly brackets in (62). In the mth term in (62) all but the p_m integral can be done trivially to yield

$$
\sqrt{\alpha(q)\alpha(q_0)}(-\frac{it}{\hbar})(2N)\Big((2m+1)\alpha_0 + (2N-2m-1)\alpha\Big)^{-2} \times
$$
$$
\int \frac{dp}{(2\pi\hbar)^n} \exp(ip_m(q-q_0)/\hbar)J(q,p_m,q_0)
$$
$$
= (-i2tN/\hbar)\int_0^\infty \beta d\beta \sqrt{\alpha(q)} \exp(-\beta(2N-2m-1)\alpha(q)) \quad \times
$$
$$
\left[\int \frac{dp}{(2\pi\hbar)^n} \exp(ip(q-q_0)/\hbar)\right) J(q,p,q_0)\right]\sqrt{\alpha(q_0)} \exp(-\beta(2m+1)\alpha_0(q))
$$

$$(65)$$

In the last step above we have dropped the subscript m from the integration variable p_m and have replaced the factor $(...)^{-2}$ by an integral over a β using the identity

$$
\frac{1}{x^2} = \int_0^\infty \beta d\beta \exp(-\beta x) \tag{66}
$$

We have also ordered the q and q_0 dependence to the left and to the right. Now noticing that $J(q,p,q_0)$ is exactly the order $\epsilon(=t$ here) term in (41) for $(\widehat{\alpha H_0})$ and proceeding in manner identical to the proof of proposition 2.3, we realize that in the action of $\mathcal{K}(qt,q_00)$ on $\psi(q_0)$ this term contributes

$$
-\left(\frac{2iNt}{\hbar}\right)\int_0^\infty \beta d\beta \sqrt{\alpha(q)} \exp[-\beta(2N-2m-1)\alpha(q)] \times (2\widehat{\alpha H_0}) \times
$$
$$
\exp[-\beta(2m+1)\alpha(q)] \sqrt{\alpha(q)}\ \psi(q) \tag{67}
$$

The factor 2, in $(2\widehat{\alpha H_0})$, in the above expression occurs because (32) was defined for $t_2 - t_1 = \epsilon_1 + \epsilon_2 = 2\epsilon$. Thus we have,

$$
\int \mathcal{K}[H_0,1,\alpha](qt,q_00)\psi(q_0)d^n q_0 = \psi(q) - \lim_{N\to\infty}\left(\frac{4itN}{\hbar}\right)\sum_{m=1}^{N-1}\widehat{I_m}\psi(q) \tag{68}
$$

where the operator $\widehat{I_m}$ is

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$$\widehat{I_m} \equiv \int_0^\infty \beta d\beta \Big\{ \alpha(q) \exp[-\beta(2N - 2m - 1)\alpha(q)] \Big\} \times$$

$$\left(\frac{1}{\sqrt{\alpha}} \, (\widehat{\alpha H_0}) \, \frac{1}{\sqrt{\alpha}} \right) \Big\{ \alpha(q) \exp[-\beta(2m + 1)\alpha(q)] \Big\} \qquad (69)$$

To do the summation over m, we first pull $\{\beta \, exp[-\beta(2m + 1)\alpha(q)]\}$ across $\alpha^{-1/2} \, (\widehat{\alpha H_0}) \, \alpha^{-1/2}$ to the left, perform β integration and then take the limit $N \to \infty$. Commuting the factor $\beta \exp[-\beta(2m + 1)\alpha(q)]$ across the operator $\alpha^{-1/2} \, (\widehat{\alpha H_0}) \, \alpha^{-1/2}$ to the left gives

$$I_m = \int_0^\infty d\beta \beta^2 \exp[-2\beta N \alpha(q)] \left(\frac{1}{\sqrt{\alpha}} (\widehat{\alpha H_0}) \frac{1}{\sqrt{\alpha}} \right) + \cdots \qquad (70)$$

where $+ \cdots$ denote the terms which do not contribute when (70) is substituted in (68) and limit $N \to \infty$ is taken. Therefore, the terms of order t in (68) take the form

$$\lim_{N \to \infty} \sum_{m=0}^{N-1} (4N) \widehat{I_m} \psi$$

$$= \lim_{N \to \infty} (4N) \sum_{m=0}^{N-1} \int_0^\infty \beta d\beta \alpha(q) \exp[-\alpha(2N - 2m - 1)\alpha] \times$$

$$(\alpha^{-1/2} \, (\widehat{\alpha H_0}) \, \alpha^{-1/2}) \alpha \exp[-\alpha(2m + 1)\alpha] \psi$$

$$= \lim_{N \to \infty} (4N) \sum_{m=0}^{N-1} \int_0^\infty \beta d\beta \alpha^2 \exp[-2\beta N \alpha(q)] \left(\alpha^{-1/2} \widehat{\alpha H_0} \alpha^{-1/2} \psi \right) + \cdots (71)$$

$$= (\alpha^{-1/2} \widehat{\alpha H_0} \alpha^{-1/2}) \psi \qquad (72)$$

where we have used (70) and

$$(\widehat{\alpha H_0}) = \frac{\partial}{\partial q^i} (\alpha g^{ij} \frac{\partial}{\partial q^j}) \qquad (73)$$

Thus we have the result

$$\int \mathcal{K}[H_0, 1, \alpha](qt, q_0 0) \psi(q_0) d^m q_0$$

$$= \psi(q) - \frac{it}{\hbar} \{\alpha^{-1/2} (\widehat{\alpha H_0}) \alpha^{-1/2}\} \psi(q) + O(t^2) \qquad (74)$$

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This above analysis could have been done with a potential term V added to the Hamiltonian H , without altering anything.

The proof of the proposition 3.1 is completed by the proving that the terms omitted in (71) do not contribute to the final answer (72). For this purpose we shall first calculate the effect of commuting the factor $\exp[-\beta(2m+1)\alpha]$ to the left across $\widehat{\alpha H}$ in (69). We note that

$$
(\widehat{\alpha H_0})\alpha \exp[-\beta(2m+1)\alpha(q)] = \beta \exp[-\beta(2m+1)\alpha(q)]\times
$$
$$
\Big[\widehat{\alpha H_0} + \{1 - \beta(2m+1)\alpha\}2g^{ij}\alpha_{ij} + \{1 - \beta(2m+1)\alpha\}(\alpha g^{ij})_{,i}\,(\alpha_{,j}\,/\alpha)
$$
$$
+\{\beta^2(2m+1)^2\alpha - (2m+1)2\beta\}g^{ij}\alpha_{,i}\,\alpha_{,j} +\{1 - \beta(2m+1)\alpha\}g^{ij}\alpha_{,ij}\Big] \quad (75)
$$

Substituting (75) in (69), the relevant integrals for us are

$$
\lim_{N\to\infty}(4N)\sum_{m=0}^{N-1}\int_0^\infty \beta d\beta \alpha^2 \exp[-2N\beta\alpha]
$$
$$
= \lim_{n\to\infty}(4N)N\alpha^2\frac{1}{(2N\alpha)^2}
$$
$$
= 1 \quad (76)
$$

$$
\lim_{n\to\infty}(4N)\sum_{m=0}^{N-1}\int_0^\infty \beta d\beta \alpha^2 \exp[-2N\beta\alpha](1 - \beta(2m+1)\alpha)
$$
$$
= 1 - \lim_{n\to\infty}4N(N^2 + O(N))\frac{2\alpha^3}{(2N\alpha)^3}
$$
$$
= 0 \quad (77)
$$

$$
\lim_{n\to\infty}(4N)\sum_{m=0}^{N-1}\int_0^\infty \beta d\beta \alpha^2 \exp[-2N\beta\alpha]\{\beta^2(2m+1)^2\alpha - 2\beta(2m+1)\}
$$
$$
= 4N\alpha^3\Big(\frac{4}{3}N^3 + O(N^2)\Big)\frac{6}{(2N\alpha)^4} - 4N\alpha^2 2(N^2 + O(N))\frac{2}{(2N\alpha)^3}
$$
$$
= 0 \quad (78)
$$

Thus all the terms in (75), except the first one, vanish when the beta integral is done and as $N \to \infty$ is taken. It is proved that the terms not written explicitly in (71) do not contribute to the final result (72).

The equation (74) gives the desired result for $\rho = 1$. The proof for a general ρ follows if use is made of the fact that the ρ dependence of the $\mathcal{K}[H, \rho, \alpha]$ is simple and is given by

$$\mathcal{K}[H, \rho, \alpha] = \frac{1}{\sqrt{\rho(q)\rho(q_0)}} \mathcal{K}[H, 1, \alpha] \qquad (79)$$

When the hamiltonians of interest is quadratic in momenta an alternative, but considerably more complicated, proof of the above propositon can be given by first converting the STP into an equivalent Lagrangian form by doing the momentum integration in the STP and converting it to the standard mid point form using the McLaughlin Schulmabn trick. Next N- fold product of this form of STP, with intermediate $N-1$ fold integrations over q variables, is substituted for the HPI1 appearing in the right hand side of (51). Repeated use of McLauglin Schulman after carefully identifying the important terms for small t and performing all the integrations again once leads to the same result in the limit $N \to \infty$. For more details on this method we refer the reader to ref 38.

It must be emphasized that though t can be taken to be small for the present purpose, we cannot take σ small and replace HPI1 in (51) by a *single* STP. A useful corollary of the proposition is obtained by taking $\alpha = \rho = \sqrt{g}$ leading to the result

Proposition 3.2 : For the hamiltonian function $H_0 = \frac{1}{2m} g^{ij} p_i p_j$, the HPI2 $\mathcal{K}[H_0, \sqrt{g}, \sqrt{g}]$, has the normalization

$$\lim_{t \to t_0} \mathcal{K}[H_0, \sqrt{g}, \sqrt{g}](qt; q_0 t_0) = g^{-1/2} \delta(q - q_0) \qquad (80)$$

and satisfies the Schrodinger equation

$$
\begin{aligned}
i\hbar \frac{\partial \psi}{\partial t} &= g^{-1/2} \, (\widehat{\sqrt{g} H_0}) \,)\psi \\
i\hbar \frac{\partial \psi}{\partial t} &= -\frac{\hbar^2}{2m} g^{-1/2} \frac{\partial}{\partial q^i} \left(g^{1/2} g^{ij} \frac{\partial}{\partial q^j} \psi \right)
\end{aligned}
\qquad (81)
$$

For later use we note that the HPI2 $\mathcal{K}[H, \sqrt{g}, \sqrt{g}]$ can also be written as

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$$\mathcal{K}[H, \sqrt{g}, \sqrt{g}](qt, q_0 0)$$
$$= \int_{-\infty}^{\infty} \frac{dE}{2\pi\hbar} \exp(-iEt/\hbar) \int_0^{\infty} d\sigma K[\alpha(H - E), 1](q\sigma, q_0 0) \qquad (82)$$

The path integral (82 was at first introduced in paper II where it was proved, using a different approach, that it satisfies the correct Schrodinger equation in arbitrary co-ordinates.

Proposition 3.3 : The HPI2 $\mathcal{K}[H, \rho, \alpha]$ is related to HPI1 without scaling by means of formula

$$\mathcal{K}[H, \rho, \alpha] = K[H - U(\alpha), \rho] \qquad (83)$$

The form of $U(\alpha)$ depends on the Hamltonian function $H(q, p)$. For $H(q, p)$ given by (42) the function $U(\alpha)$ is defined by (49). The above result is proved by showing that both sides have the same normalization and satisfy the same Schrodinger equation.

Proposition 3.4 : For a trivial scaling when the scaling function $\alpha(q) =$ constant, say, c the HPI2 $\mathcal{K}[H, \rho, c]$ is equal to the HPI1 $K[H, \rho]$ without scaling.

$$\mathcal{K}[H, \rho, c] = K[H, \rho] \qquad (84)$$

This equation with $\alpha =$ constant is easily proved by following the proof of the proposition 3.1 upto equation (62) and noticing that all the terms involving the derivatives of $\alpha(q)$ vanish and thus (62) leads to the discrete form (35).

4. An example of quantization in arbitrary coordinates

The equation (81) is recognised as the correct Schrodinger equation in arbitrary co-ordinates. This makes HPI2 $\mathcal{K}$ useful for quantization in arbitrary coordinates. In some special cases $\mathcal{K}$ itself is equal to the correct propagator in arbitrary cordinates. In general situations, more frequently encountered, the propagator can be easily written in terms of the HPI2 $\mathcal{K}$ defined above. As an example of quantization in arbitrary coordinates and a simple example of exact path integration we set up the path integral quantization of $1/r$ potential in parabolic coordinates in two dimensions obtain the exact answer for the Green function.

The classical Hamiltonian for the two dimensional Coulomb problem is

$$H = \frac{\vec{p}^2}{2m} - \frac{e^2}{r} \tag{85}$$

In the two dimensional parabolic coordinates, u_1 and u_2, defined by

$$x = u_1^2 - u_2^2 \quad , \quad y = 2u_1u_2 \tag{86}$$
$$dxdy = 4\vec{u}^2 du_1 du_2 \quad , \quad g^{1/2} = 4u^2 = u_1^2 + u_2^2 \tag{87}$$

the classical hamiltonian takes the form

$$H = \frac{\vec{p}_u^2}{8mu^2} - \frac{e^2}{u^2} \tag{88}$$

Quantization in parabolic coordinates u_1, u_2 will proceed via path integral HPI2 with hamiltonian H, scaling function $\alpha \equiv 4\vec{u}^2$. Therefore we consider the hamiltonian path integral of the second kind

$$\mathcal{K}[H, 4u^2, 4u^2](\vec{u}t; \vec{u}_0 0) = \int \frac{dE}{(2\pi\hbar)} \exp(-iEt/\hbar) \int_0^\infty d\sigma K[H_E, 1](\vec{u}\sigma; \vec{u}_0 0) \tag{89}$$

with

$$H_E = 4u^2(H - E) = \left(\frac{\vec{p}_u^2}{2m} - 4e^2 - 4Eu^2 \right) \tag{90}$$

Notice that apart from an additive constant, H_E is just the harmonic oscillator hamiltonian in two dimensions. Thus we see that the HPI1 in (89) is related to the propagator for the two dimensional oscillator and the relation is given by

$$K[H_E, \rho = 1](\vec{u}\sigma; \vec{u}_0 0) = \exp(i4e^2\sigma/\hbar)\mathbf{k}^{osc}\langle u_1\sigma|u_{10}0\rangle \mathbf{k}^{osc}\langle u_2\sigma|u_{20}0\rangle \tag{91}$$

with $\mathbf{k}^{osc}$ is the one dimensional oscillator propagator for mass $\mu = m$ and frequency $\omega^2 = -8E/M$. Though HPI2 of (89) satisfies the correct Schrodinger equation, it is still not equal to the desired propagator. This is because at $t = 0$ it reduces to $g^{-1/2}\delta(\vec{u} - \vec{u}_0)$ where as the correct propagator at $t = 0$ becomes

$$\delta(\vec{r} - \vec{r}_0) = g^{-1/2}\left[\delta(\vec{u} - \vec{u}_0) + \delta(\vec{u} + \vec{u}_0) \right] \tag{92}$$

It is very easy to take care of this difference and the correct propagator can be written down as

$$\langle \vec{r}t|\vec{r}_0 0\rangle = [\mathcal{K}[H_0, 4u^2, 4u^2](\vec{u}t; \vec{u}_0 0) + \mathcal{K}[H_0, 4u^2, 4u^2](\vec{u}t; \vec{u}_0 0)] \qquad (93)$$

Thus the desired quantum mechanical propagator $\langle \vec{r}t|\vec{r}_0 0\rangle$ is obtained from (93) and using the expressions (89) and (91). Writing the answer in terms of the energy dependent Green function we get

$$\mathbf{G}(\vec{r},\vec{r}_0|E) = \int_0^\infty d\sigma \exp(4ie^2\sigma/\hbar)[\mathbf{k}_1^{osc}\langle u_1\sigma|u_{10}0\rangle\mathbf{k}_1^{osc}\langle u_2\sigma|u_{20}0\rangle$$
$$+\mathbf{k}_1^{osc}\langle u_1\sigma|-u_{10}0\rangle\mathbf{k}_1^{osc}\langle u_2\sigma|-u_20\rangle] \qquad (94)$$

After the integration variable σ is replaced by $\tau = 4\sigma$, this result agrees with the result of ref. 13.

5. Concluding remarks

In this paper we have discussed properties of two types of path integrals HPI1 and HPI2. The path integral HPI2 intoduced here is a generalization of HPI1 introduced in paper I. An importatnt result obtained in this paper is the scaling formula of proposition 3.3. We have not followed the approaches existing in the literature to derive the scaling formula or a similar result in the path integral or the operator formalism. The propagators appearing in the two sides of this formula were constructed using path integration with the same prescription for STP. We have then derived the effective potential that the scaling formula introduces in the classical Lagrangian or Hamiltonians. The method used by us is a direct approach starting from the definintions used and can also be used for Lagrangian path integrals.

Our use of the path integral with scaling also differs from the use in the existing literature where local scaling is merely one of the techniques to be used in exact path integration. For us the local scaling is an essential ingredient for quantization by means of hamiltonian path integration in arbitrary coordinates. Use of local scaling in the hamiltonian path integral scheme enables us to formulate path integral quantization scheme in terms of the classical hamiltonian only, that no $O(\hbar^2)$ terms are needed is a unique feature of this scheme. At present the origin of nontrivial scaling necessary in the hamiltonian formalism for non-cartesian coordinates is not understood from any deeper physical or mathematical reason.

We believe that the Hamiltonian path integral formalism can be used to discuss everything, and at the same level of rigor, as in the Lagrangian approach. To support this claim we should demonstrate the applicability of the Hamiltonian method to exact path integration of various potential prob-

lems. In the existing literature there are other techniques, notably, the trick of adding new degrees of freedom, which also play an important role in exact path integral evaluations. However, these are best discussed for each specific case of exact path integration. In this paper we have restricted ourselves to the local rescaling of time only because it is the general idea common to all such calculations. A discussion of other techniques together with exact path integration of potential problems will be taken up in a separate publication[38].

It is not just that it may be possible to do everything in the hamiltonian approach that can be done in existing Lagrangian formalism; the Hamiltonian approach offers distinct advantages in certain problems. As is well known the constraint analysis and unitarity are most transparent in phase space path integral formalism, and it should be possible to give a time slicing definition of the formal path integrals of Faddeev[39] and of Senjanovic [40] for the constrained systems.

To elaborate on the Hamiltonian path integral with local scaling of time and show its efficacy in handling quantization in arbitrary co- ordinates has been the main objective of this paper. In the literature the hamiltonian path integral scheme have not received as much attention as the Lagrangian path integral approach, which incidentally, is reflected in the sketchy treatment given to the canonical path integral in books such as by Schulman[7]. We fondly hope that, with our presentation here, the impression that the Hamiltonian path integral is meant only for a formal discussion wil be removed.

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